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A Linear Superposition Formula for the Sine-Gordon Multisoliton Solutions

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We present simple constructions of the multisoliton solution of the sine-Gordon equation (i) as a nonlinear combination of asymptotic (i.e. constant speed) solitons and (ii) as a linear superposition of accelerating kinks. We argue that the latter provides a more consistent interpretation of a multisoliton solution as an interaction between its asymptotic components.

§1. Introduction

The sine-Gordon equation (SGE)

$$\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial t^2}=\sin\phi, \tag{1}$$

where ϕ: $\boldsymbol{R}^2\to\boldsymbol{R}$, is the equation of motion of a single, dimensionless scalar field in one space and one time dimension. The equation is clearly invariant under Lorentz transformations as well as translations in space and time. In addition it possesses the discrete symmetries $x\to -x$, $t\to -t$ and $\phi\to 2n\pi\pm\phi$, $n\in\boldsymbol{Z}$, where the last symmetry is taken into account by restricting the range of the real-valued map ϕ to the unit circle $[0, 2\pi]$.

Now it is well known that the SGE has a special class of analytic solutions—the multisolitons, and that explicit expressions for these solutions can be obtained by using the inverse scattering formalism.[1-3] Another representation of these multisolitons is found by using the Bäcklund transformation of the SGE.[4] This is a 'ladder' method in which one soliton is added at each application of the transformation, the resulting expressions often being referred to as nonlinear superposition formulas. For example: let ϕ_1 be a soliton and ϕ_{21}, ϕ_{22} be the two-solitons generated from ϕ_1 by Bäcklund parameters a_1, a_2 respectively. Then a three-soliton solution ϕ_3 is given by the formula

$$\phi_3=\phi_1+4\tan^{-1}\left[\left(\frac{a_1+a_2}{a_1-a_2}\right)\tan\left[\frac{1}{4}(\phi_{21}-\phi_{22})\right]\right].$$

Remarks: (1) The superposition formula is entirely in terms of soliton functions.

(2) The extension of the formula to general N can be found in a paper by Barnard.[5]

Although the inverse scattering and Bäcklund methods lead to explicit expressions for multisoliton solutions, these expressions are complicated and it is difficult to see how the whole is constructed from its parts, i.e. how the individual solitons are glued together to form the multisolitons. Furthermore, the concept of "nonlinear superposition" is, to say the least, functionally ambiguous. This raises two interesting questions about the multisoliton solutions of the SGE. Firstly, is there a representation of solutions which is simple to construct and in which the individual solitons are clearly identifiable? Secondly, can the solutions be represented as a linear superposition of some characteristic functions? In view of the many remarkable properties of the SGE, it is not surprising that the answer to both these questions is yes. A simple algorithm for the construction of N-soliton solutions was originally proposed by Bowtell and Stuart[6] in their investigation of the particle-like nature of SGE solitons. They showed, by direct transformations from the Hirota formula,[7] that the algorithm reproduced the correct solutions for the cases $N=2$ and $N=3$ and conjectured that it should work for any N. A linear superposition formula for the case $N=2$, which is implicit in the work of Bowtell and Stuart, was first published in explicit form by Matsuda.[8]

In this paper we give a rigorous proof of the Bowtell-Stuart conjecture and show that it leads directly to a linear superposition formula for the multisoliton solutions of the SGE. Unlike the Bäcklund method, which requires the solution of a pair of coupled first order partial differential equations, our proofs are entirely algebraic.

§2. General Formalism

To make the paper self-contained and to establish our notation we start with a brief review of some basic definitions and results.

The single solitons $\phi:(x, t, u, \alpha)\rightarrow(0, 2\pi)$ are solitary wave solutions of (1) defined by the functions

$$\tan\left(\frac{1}{4}\phi\right)=\exp\left[\varepsilon\gamma(x-ut)+\alpha\right], \qquad (2)$$

where $|u|<1$ is the speed of a wave, $\alpha\in\boldsymbol{R}$ its phase and $\gamma=(1-u^2)^{-1/2}\geqslant 1$ the Lorentz contraction factor. $\varepsilon=+1$ corresponds to what is conventionally called the soliton and $\varepsilon=-1$ to the antisoliton. Thus, the single solitons are a set of two-parameters maps from $\boldsymbol{R}^2$ onto $(0, 2\pi)$ and any element of this set is identified by its velocity, phase and signature. It is clear that ϕ is an analytic function of all its variables.

The N-solitons, Ψ_N, are $2N$-parameter maps from R^2 onto $(0, 2\pi)$ with the property that their asymptotic values as $|t|\rightarrow\infty$ are the same as those of a linear superposition of N single solitons. Specifically, if we let $\xi=\gamma_i(x-u_it)$ and $\eta=\gamma_i(t-u_ix)$ then there exists a set of N distinct values of u_i such that $\lim_{\eta\rightarrow\pm\infty}\Psi_N(\xi, \eta)=\phi_i(\xi, u_i, \alpha_{i\pm})$, where α_{i-}, the phase at $t=-\infty$, is in general different from α_{i+}, the phase at $t=+\infty$. (Cf. Wadati and Toda[9] where this definition is implemented for the Korteweg-de Vries equation.)

The inverse scattering formalism now implies the following result. Let S_N be the set of N-soliton functions and S_1^N the cartesian product of N copies of S_1, then, for each $N\in\boldsymbol{Z}^+$ there exists a map $\mathrm{F}:S_1^N\rightarrow S_N$. A specific representation which we shall use is expressed by the following theorem:

Theorem 1 Let $(\phi_1,\cdots,\phi_N)\in S_1^N$, where each ϕ_i is labelled by its velocity, phase and signature, i.e. $(u_i, \alpha_i, \varepsilon_i)$, with $u_i\neq u_j$. Then $F(\phi_1,\cdots,\phi_N)$ is the multisoliton solution ψ_N given by

$$\psi_N=-4\tan^{-1}\left[\frac{\mathit{Im}\det(I-iM)}{\mathit{Re}\det(I-iM)}\right], \qquad (3a)$$

where M is the $N\times N$ matrix with elements

$$m_{ij}=\frac{2a_j}{(\lambda_{ij}a_i+a_j)}\exp\left[\frac{1}{2}(\varepsilon_ia_i+\varepsilon_ja_j)\xi+\eta/(\varepsilon_ja_j)+\alpha_j\right], \qquad (3b)$$

$a_i=[(1+u_i)/(1-u_i)]^{1/2}$ is an element of $\boldsymbol{R}^+$, $\varepsilon_i=\pm1$, $\lambda_{ij}=\varepsilon_i\varepsilon_j$ and $\xi=\frac{1}{2}(x-t)$, $\eta=\frac{1}{2}(x+t)$ are characteristic coordinates. □

Remarks: (1) This is a standard result[2] and full details of the proof of the theorem can be found in ref. 4.

(2) The ε_i, $i=1,\cdots,N$, must be incorporated into the scattering data in order for the inverse map of the scattering transform to have a unique image.

§3. Construction of the Multisoliton Solutions

3.1 The nonlinear algorithm

In this section we turn to the statement and proof of the theorem implied by the Bowtell-Stuart algorithm for a multisoliton solution of the SGE. The data necessary for this construction is, firstly, the set of solitons $\{\phi_i\}_{i=1}^N$ defined above, and, secondly, the set $\{u_{ij}\}$, where u_{ij} is the common speed of the ith and jth solitons in their centre of velocity frame.

Lemma 1 $u_{ij}\in(0, 1)$ is a Lorentz invariant functional on the two solitons ϕ_i, ϕ_j and is given by the expressions

$$u_{ij}=\frac{\gamma_i\gamma_j|u_i-u_j|}{1+\gamma_i\gamma_j(1-u_iu_j)}=\frac{|a_i-a_j|}{(a_i+a_j)}, \tag{4}$$

where the a_i are defined in Theorem 1. □

Theorem 2 Let $(\phi_1,\cdots,\phi_N)$, where $\phi_i\in S_1$, be an N-tuple of solitons and antisolitons with distinct velocities. Define the maps g and h as follows:

$$g\colon S_1^N \longrightarrow T_N,\quad g(\phi_1,\cdots,\phi_N)=\tan\left[\sum_{i=1}^{N}\frac{1}{4}\phi_i\right]=\sigma,$$

$$h\colon T_N \longrightarrow W_N\colon \sigma \longrightarrow w,$$

where w is obtained from σ by scaling each pairwise product of tans as $\tan(\frac{1}{4}\phi_i)\tan(\frac{1}{4}\phi_j)\to(u_{ij})^{2\lambda_{ij}}\tan(\frac{1}{4}\phi_i)\tan(\frac{1}{4}\phi_j)$. Then the composite map $4\tan^{-1}[(hog)(\phi_1,\cdots,\phi_N)]$ is a multisoliton solution, i.e. $W_N=S_N$. □

Example In the case of three solitons, i.e., $\varepsilon_1=\varepsilon_2=\varepsilon_3=+1$, we have

$$g(\phi_1,\phi_2,\phi_3)=\tan\left[\frac{1}{4}(\phi_1+\phi_2+\phi_3)\right]=\frac{T_1+T_2+T_3-T_1T_2T_3}{1-T_1T_2-T_1T_3-T_2T_3},$$

where $T_i=\tan(1/4\,\phi_i)$, and

$$(hog)(\phi_1,\phi_2,\phi_3)=\frac{T_1+T_2+T_3-(u_{12}u_{13}u_{23})^2T_1T_2T_3}{1-u_{12}^2T_1T_2-u_{13}^2T_1T_3-u_{23}^2T_2T_3}=\tan\left(\frac{1}{4}\psi_3\right),$$

giving the three-soliton solution ψ_3.

Proof This consists of showing that the inverse scattering result of Theorem 1 above can be reduced to the form required by our theorem. In our proof, and in what follows, we simplify the notation by writing $v_{ij}=(u_{ij})^{\lambda_{ij}}$.

We start with the characteristic polynomial of the matrix iM defined in Theorem 1 which is given by

$$\det(iM-\lambda I)=\sum_{r=0}^{N} i^{N+r}\mu_{N-r}\lambda^r,$$

where μ_r is the sum of the principal minors of order r of $\det M$ with the convention $\mu_0=1$. Thus, the inverse scattering result (3a) for the multisoliton solution can be written as

$$\tan\left[\frac{1}{4}\psi_N\right]=\frac{\sum_{r=0}^{n_1}(-1)^r\mu_{2r+1}}{1+\sum_{r=1}^{n_2}(-1)^r\mu_{2r}}, \tag{5}$$

where $n_1=[\frac{1}{2}(N-1)]$ and $n_2=[\frac{1}{2}N]$, with [.] denoting integer part of.

Now the symmetry of M implies that the principal minors of M have the same form as the determinant of M. Thus, all that we need is an explicit formula for the determinant of M itself and this will determine the μ_r in the multisoliton expression (5).

It is easy to verify that M factorises as follows:

$$M=B(\xi)AB(\xi)C(\eta),$$

where A is the $N\times N$ matrix with elements $a_{ij}=2a_j/(\lambda_{ij}a_i+a_j)$, B is the diagonal matrix with elements $b_{ij}=\delta_{ij}\exp(\frac{1}{2}\varepsilon_ja_j\xi)$, C is the diagonal matrix with elements $c_{ij}=\delta_{ij}\exp(\eta/\varepsilon_ja_j+\alpha_j)$. Thus, using the notation given in Theorem 1, the expression for $\det M$ in laboratory coordinates is given by

$$\det M=\det A\prod_{i=1}^{N}a_i\exp[\varepsilon_i\gamma_i(x-u_it)+\alpha_i].$$

Turning now to $\det A$, we write

$$\det A\prod_{i=1}^{N}a_i=\det H_N,$$

where H_N is the symmetric $N\times N$ matrix with elements $h_{ij}=2(a_ia_j)^{1/2}/(\lambda_{ij}a_i+a_j)$. The row operations row $i\to$row $i-h_{iN}$ row N $(i=1,\cdots,N-1)$ transform H_N to $\bar{H}_N$ where

$$\bar{h}_{ij}=\left[\frac{a_N-\lambda_{Nj}a_j}{a_N+\lambda_{Nj}a_j}\right]\left[\frac{a_N-\lambda_{Ni}a_i}{a_N+\lambda_{Ni}a_i}\right]h_{ij},$$
$$(i,j=1,\cdots,N-1),$$

$\bar{h}_{iN}=0$, $(i=1,\cdots,N-1)$ and $\bar{h}_{Nj}=h_{Nj}$ $(j=1,\cdots,N)$. Extracting row and column factors and noting from the expression for u_{ij} in Lemma 1 that

$$(u_{iN})^{2\lambda_{iN}}=v_{iN}^2=\left[\frac{a_N-\lambda_{iN}a_i}{a_N+\lambda_{iN}a_i}\right]^2 \text{ we obtain}$$

$$\det H_N=\det H_{N-1}\prod_{i=1}^{N-1} v_{iN}^2.$$

Successive applications of this result give

$$\det H_N=\prod_{i=1}^{N-1}\prod_{j=i+1}^{N} v_{ij}^2.$$

Hence,

$$\det M=\prod_{i=1}^{N-1}\prod_{j=i+1}^{N} v_{ij}^2\prod_{k=1}^{N}\exp\left[\varepsilon_k\gamma_k(x-u_kt)+\alpha_k\right].$$

We can now read off the μ_r and these are given by

$$\mu_0=1,$$

$$\mu_1=\sum_{i=1}^{N} T_i,$$

$$\mu_r=\sum\prod_{i=1}^{r} T_i\prod_{j=i+1}^{r} v_{ij}^2,\quad (r>1),$$

where the expression behind the summation sign in μ_r is the first principal minor and the other principal minors are obtained by taking all combinations of r integers from $1,\cdots,N$ in place of $1,\cdots,r$. For example, for $N=4$ with $\varepsilon_1=\varepsilon_2=+1$ and $\varepsilon_3=\varepsilon_4=-1$,

$$\mu_3=u_{12}^2(u_{13}u_{23})^{-2}T_1T_2T_3+u_{12}^2(u_{14}u_{24})^{-2}T_1T_2T_4$$
$$+u_{13}^2(u_{14}u_{34})^{-2}T_1T_3T_4$$
$$+(u_{23}u_{24})^{-2}u_{34}^2T_2T_3T_4.$$

Clearly, by putting all multipliers $u_{ij}=1$, the expression (5) for the N-soliton solution reduces to $\tan\left[\frac{1}{4}\sum_{i=1}^{N}\phi_i\right]$. Hence, with the multipliers included, ψ_N is just the composite function $(\mathrm{hog})(\phi_1,\cdots,\phi_N)$ of the theorem.

In the case of the 3-soliton example given after the statement of the theorem, we have

$$\mu_1=T_1+T_2+T_3,$$
$$\mu_2=u_{12}^2T_1T_2+u_{13}^2T_1T_3+u_{23}^2T_2T_3,$$
$$\mu_3=(u_{12}u_{13}u_{23})^2T_1T_2T_3,$$

which when substituted in (5) gives the formula stated in the example.

Apart from being simple to construct, the form of the multisoliton solution given in the theorem above leads to a natural interpretation of the interactions between its component solitons. Thus, consider the two-soliton solution. A linear superposition of these solitons is given by $\psi_2^L=\sum_{i=1}^2\phi_i$, $\varepsilon_1=\varepsilon_2=+1$, which implies that

$$\tan\left(\frac{1}{4}\psi_2^L\right)=\frac{\sum_{i=1}^{2}\tan\left(\frac{1}{4}\phi_i\right)}{1-\prod_{i=1}^{2}\tan\left(\frac{1}{4}\phi_i\right)}.$$

This represents the profile of two free, i.e., noninteracting, solitons and is the asymptotic, i.e. $|t|\to\infty$, form of the two-soliton solution modulo phases. The interaction between these solitons is "switched on" by introducing the invariant u_{12}^2 into the product term in the denominator giving

$$\tan\left(\frac{1}{4}\psi_2\right)=\frac{\sum_{i=1}^{2}\tan\left(\frac{1}{4}\phi_i\right)}{1-u_{12}^2\prod_{i=1}^{2}\tan\left(\frac{1}{4}\phi_i\right)},$$

which is the two-soliton solution for finite values of t.

We now generate the N-soliton solution $(N>2)$ from an asymptotic state of N free solitons and antisolitons by forming a linear superposition of the single-soliton functions, i.e.

$$\tan\left(\frac{1}{4}\psi_N^L\right)=\tan\left(\frac{1}{4}\sum_{i=1}^{N}\phi_i\right),$$

and then switching on the interactions between all pairs by introducing the two-soliton invariants $(u_{ij})^{2\lambda_{ij}}$. Thus, the multisoliton interaction may be thought of as a combination of two soliton interactions and hence is qualitatively similar to the N-body interactions in Newtonian graviation.

3.2 *The linear superposition form*

The interpretation given above, although it allows us to describe the multisoliton solution as a set of pairwise interactions, still has one major drawback when compared with the corresponding particle problem. This is that the individual solitons in a multisoliton solution can only be identified in the noninteracting asymptotic limits $t \to \pm\infty$. For all finite times their identities are lost in the nonlinearity of the multisoliton formula. In order to identify the solitons throughout the interaction process, it is necessary to have the solution in a form which is manifestly separable into N component functions at all finite times and not just asymptotically. Now one way of achieving this objective is to attempt to pull-back the linear superposition form from $|t|=\infty$ to finite t. That this can be done successfully is the content of the next theorem.

Theorem 3 Let ψ_N be the multisoliton solution of the SGE as constructed in Theorem 2. Then there exist N real analytic functions $f_i(x, t)$, $(i=1, \cdots, N)$, such that

(i) $\psi_N(x, t)=4\sum_{i=1}^{N}\tan^{-1}[f_i(x, t)]$,

(ii) in the rest frame of ϕ_i,

$\lim_{|t|\to\infty} 4\tan^{-1}[f_i(x, t)]=\phi_i(x)$. □

Proof Comparing the expansion

$$\tan\left[\frac{1}{4}\psi_N\right]=\tan\left(\sum_{i=1}^{N}\tan^{-1}f_i\right),$$

with the expression for ψ_N given in Theorem 2, it can be seen, using the notation given before, that they will be the same if the f_i satisfy the following identities:

$$\sum_{i=1}^{N} f_i=\sum_{i=1}^{N} T_i=A_1, \tag{6a}$$

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} f_i f_j=\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} v_{ij}^2 T_i T_j=A_2, \tag{6b}$$

$$\sum\prod_{i=1}^{r} f_i=\sum\prod_{i=1}^{r} T_i\prod_{j=i+1}^{r} v_{ij}^2=A_r, \quad (1<r\leqslant N), \tag{6c}$$

where the summation in the last identity indicates taking all combinations of r integers from $1, \cdots, N$ in place of $1, \cdots, r$. From the form of the left hand sides of the above expressions defining A_r, it follows that the f_i are the roots of the polynomial

$$f^N-A_1 f^{N-1}+A_2 f^{N-2}-\cdots+(-1)^N A_N=0.$$

Now consider the matrix P with elements

$$p_{ij}=[(1-v_{ij}^2)T_i T_j]^{\frac{1}{2}}.$$

Noting that

$$P=\operatorname{diag}(\sqrt{T_1}, \cdots, \sqrt{T_N})H_N \times \operatorname{diag}(\sqrt{T_1}, \cdots, \sqrt{T_N}),$$

where H_N is the matrix defined in the proof of the previous theorem, we have that

$$\det P=\det M=\prod_{i=1}^{N} T_i\prod_{j=i+1}^{N} v_{ij}^2.$$

The symmetry of P implies that the principal minors of P have the same form as det P and hence it is easy to see, from the identities (6) defining A_r above, that A_r is the sum of the principal minors of order r of P. Thus, the polynomial in f is the characteristic polynomial of P, i.e., the $f_i(x, t)$ are the eigenvalues of P. Since P is real and symmetric, it has N real eigenvalues. Furthermore, since P is an analytic matrix, its eigenvalues are analytic functions. This proves the first part of the theorem.

For the second part of the theorem, we first note the property that the symmetry $\phi_i\to\phi_i\pm 2\pi$ implies that the composition map (hog)$(\phi_1, \cdots, \phi_N)$ is invariant under the substitution $T_i\to -1/T_i=\bar{T}_i$.

Let us assume that the velocities u_i are relabelled such that $u_i<u_{i+1}$, $(i=1, \cdots, N-1)$. The argument is then as follows:

(1) Go to the rest frame (ξ, η) of ϕ_r by means of the Lorentz transformation $x=\gamma_r(\xi+u_r\eta)$, $t=\gamma_r(\eta+u_r\xi)$. This transforms T_i to $T_i=\exp[\varepsilon_i\Gamma_{ir}(\xi-U_{ir}\eta)]$, $(i\neq r)$, where $U_{ir}=(u_i-u_r)/(1-u_i u_r)$ and $\Gamma_{ir}=(1-U_{ir}^2)^{-1/2}$ and T_r to $T_r=\exp[\varepsilon_r\xi]$.

(2) The symmetry $T_i\to\bar{T}_i$ can now be used in the identities (6) to ensure that the coefficient of η in the index of each exponential is negative for all $i\neq r$. Explicitly the rule is as follows: if $\varepsilon_i U_{ir}>0$ then T_i is left unchanged, whereas if $\varepsilon_i U_{ir}<0$ then T_i is replaced with $\bar{T}_i$.

(3) We now proceed to the limit $\eta=\infty$ when the identities (6) assume the asymptotic forms

$$\sum_{i=1}^{N} f_i=T_r,$$

$$\sum\prod_{i=1}^{r} f_i=0, \quad (1<r\leqslant N).$$

These equations are satisfied if and only if every f except one is zero. We call the nonzero one f_r. Clearly the proof of the theorem can be readily adapted to the case of $\eta \to -\infty$. Hence we have

$$\lim_{|t|\to\infty} 4\tan^{-1}[f_r(x,t)] = \phi_r(x),$$

and this completes the proof of the theorem.

Note: The ϕ_i symmetry reappears in the linear superposition form as follows: the discrete symmetry $\psi_N \to \psi_N \pm 2n\pi$, $n \in Z$, implies that the formula is invariant under the substitution $f_i \to -1/f_i = \bar{f}_i$, in n of the N terms.

Corollary (i) The f_i are all positive functions.

(ii) The f_i are functionally distinct.

Proof (i) Since the A_r in the theorem are positive for all x and t, i.e. the polynomial in f has alternating signs, it follows by Descartes rule of signs that all the f_i are positive for all x and t.

(ii) Since the f_i are analytic functions and are asymptotically distinct, it follows that they cannot be equal on any open set in the (x, t)-domain and hence are functionally distinct for all x and t.

§4. Interpretations of the Linear Form

The linear form of the multisoliton solution given in Theorem 3 leads us, quite naturally, to identify the component solitons in the interaction with the functions $4\tan^{-1}f_i$ rather than with the ϕ_i. As we shall see below, for $i>1$ the $4\tan^{-1}f_i$ are accelerating kinks which, as shown above, tend asymptotically to the constant velocity forms ϕ_i.

Now if we want the solitons in a multisoliton solution to have individual identities during the interaction process (i.e., at finite t) so that the concept of soliton is more than just an asymptotic identification, then it follows that any particular soliton will be subject to forces exerted on it by the other solitons in the solution. We would therefore expect it to have a variable speed and thus it seems more consistent to represent SGE solitons by the functions $4\tan^{-1}f_i$ rather than by the constant velocity forms ϕ_i. It is only in the case of single solitons and asymptotic forms that the ϕ_i form a valid representation when, of course, they are equal to the $4\tan^{-1}f_i$.

Unfortunately, it seems that the only way of solving the polynomial for general N, to obtain explicit information about the functions f_i, is by numerical methods. However, in the case $N=2$, it is quite straightforward to obtain explicit formulas for these functions in the centre-of-velocity frame, i.e. $0<u_1=-u_2=u=u_{12}$, $v=(u_{12})^{\lambda_{12}}$. In this case we have to solve the quadratic equation

$$f^2-(e^{x_1}+e^{x_2})f+v^2e^{x_1+x_2}=0,$$

where $x_1=\varepsilon_1\gamma(x-ut)+\alpha_1$ and $x_2=\varepsilon_2\gamma(x+ut)+\alpha_2$. Thus

$$f_\pm=\frac{1}{2}\left[e^{x_1}+e^{x_2}\pm[(e^{x_1}+e^{x_2})^2-4v^2e^{x_1+x_2}]^{1/2}\right],$$

choosing the case $\varepsilon_1=\varepsilon_2=+1$ with $\alpha_1=\alpha_2=-\ln u$, we have, from Theorem 3, that

$$\psi_2=4\tan^{-1}\exp[\gamma(x+\sigma(t))]+4\tan^{-1}\exp[\gamma(x-\sigma(t))], \tag{7}$$

where

$$\sigma(t)=(1/\gamma)\ln\{[\cosh\gamma ut+(\cosh^2\gamma ut-u^2)^{\frac{1}{2}}]/u\}.$$

Combining the terms on the right hand side of (7) and using the symmetry transformation $\psi_2\to\psi_2+2\pi$ reduces it to the usual form

$$\psi_2=4\tan^{-1}\left[\frac{u\sinh\gamma x}{\cosh\gamma ut}\right],$$

which verifies the first part of Theorem 3.

The asymptotic expressions for ψ_2 are given by

(i) $\psi_2\sim4\tan^{-1}\{\exp[\gamma(x-ut)-\delta]\}+4\tan^{-1}\{\exp[\gamma(x+ut)+\delta]\}$, (8a) in a neighbourhood of $t=-\infty$

(ii) $\psi_2 \sim 4\tan^{-1}\{\exp[\gamma(x+ut)-\delta]\}+4\tan^{-1}\{\exp[\gamma(x-ut)+\delta]\}$, (8b)

in a neighbourhood of $t=\infty$, where $\delta=\ln u$, which verifies the second part of Theorem 3.

Clearly each term in ψ_2 in (7) is a kink and since $\sigma(t)$ is nonlinear these kinks have variable speeds. Thus, ψ_2 is a linear superposition of accelerating solitons. However, these solitons are not solutions of the SGE for finite t, but become solutions in the asymptotic limits as $t\to\pm\infty$ (eqs. 8(a) and 8(b)). The identification of these asymptotic limits in 8(a) and 8(b) is now made as follows:

$$\begin{aligned} \text{at } t=-\infty,\quad & f_1 \longrightarrow \exp[\gamma(x-ut)-\delta]=\phi_1, \\ & f_2 \longrightarrow \exp[\gamma(x+ut)+\delta]=\phi_2, \\ \text{at } t=+\infty,\quad & f_1 \longrightarrow \exp[\gamma(x+ut)-\delta]=\phi_2, \\ & f_2 \longrightarrow \exp[\gamma(x-ut)+\delta]=\phi_1. \end{aligned}$$

Thus, notice that although the f's have the same phase throughout, their asymptotic identification in terms of the constant velocity solitons leads to the phase shifts 2δ for ϕ_1 and -2δ for ϕ_2. This implements Theorem 3 in the laboratory frame.

§5. Concluding Remarks

In this paper we have presented two algorithms for the construction of multisoliton solutions of the SGE. (i) As a simple nonlinear combination of N constant velocity solitons and antisolitons ϕ_i coupled together by the invariants v_{ij} and (ii) as a linear combination of N variable speed kinks $4\tan^{-1}f_i$, where the f_i are solutions of an Nth degree polynomial. We have also argued that if a multisoliton is interpreted as an interaction between its asymptotic components, then, even though the variable speed kinks are not solutions of the SGE for finite values of time, they provide a more consistent way of identifying individual components both during the interaction process as well as asymptotically.

Another interesting property of these variable speed solitons is that their motions are correlated with the motions of the poles of the corresponding multisoliton solution and hence, as shown by Bowtell and Stuart[6] for the case $N=2$, lead to a consistent interpretation of the multisoliton solution as a system of N interacting particles. Numerical results for these particle problems in the cases $N=3$ and 4 have been obtained by us and will be published elsewhere.

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