Superposition Formulae for Sine-Gordon Multisolitons(*)

A. C. Bryan, J. F. Miller and A. E. G. Stuart

Department of Mathematics, The City University, Northampton Square
London ECIV OHB, England

(ricevuto il 10 Novembre 1987)

Summary. — We present nonlinear and linear superposition formulae for multisoliton solutions of the sine-Gordon equation. These formulae are constructed for the general case, that is for multisolitons which contain any combination of solitons, antisolitons and breathers. The nonlinear superposition is constructed from these components, whereas the linear superposition is in terms of accelerating and oscillating kinks. We argue that the latter provide a more consistent interpretation of a multisoliton solution as an interaction between its asymptotic components.

PACS 03.50.Kk – Other special classical field theories.

1. – Introduction.

One of the many interesting features of the sine-Gordon equation (SGE) is the existence of several representations of its multisoliton solutions. These distinct forms of the multisoliton functions are obtained by applying different methods to solve the equation in its soliton sector. The standard techniques used in this context are 1) the inverse scattering transform, 2) the Hirota algorithm and 3) the Bäcklund recurrence formula (1). In each case the method is constructive and leads to an explicit expression for the solution. Of course these representations are equivalent since the multisoliton solutions are unique. On

(*) To speed up publication, the authors of this paper have agreed to not receive the proofs for correction.
the other hand, in considering a specific expression for a multisoliton, with perhaps a view to applying it, it is clearly advantageous to have one which is simple to construct and whose structure is straightforward to analyse and interpret.

Unfortunately, the expressions obtained by using the methods mentioned above have rather complicated structures which, for example, makes it difficult to see how a multisoliton can be constructed and interpreted as an interaction between its elementary components, i.e. the solitons, antisolitons and breathers. However, in their investigation of the particle-like nature of sine-Gordon solitons, Bowtell and Stuart (1) showed that the Hirota formulae for the two- and three-soliton solutions could be transformed to much simpler forms. For example, in the three-soliton case, the transformed solution \( \Phi_3 \) is given by

\[
\tan \left( \frac{\Phi_3}{4} \right) = \frac{\sum_{i=1}^{3} \tan \left( \frac{\phi_i}{4} \right) - (u_{12} u_{23} u_{31}) \prod_{i=1}^{2} \tan \left( \frac{\phi_i}{4} \right)}{1 - \sum_{i=1}^{2} \sum_{j=i+1}^{3} u_{ij}^4 \tan \left( \frac{\phi_i}{4} \right) \tan \left( \frac{\phi_j}{4} \right)},
\]

where \( \phi_i, i = 1, 2, 3 \), are single solitons with distinct speeds and \( u_{ij} \) is the common speed of the \( i \)-th and \( j \)-th solitons in their centre-of-mass frame. This allowed them to interpret the three-soliton solution as a combination of two-soliton interactions determined by the (Lorentz invariant) coupling constants \( u_{ij} \) in the following manner: consider the linear sum of three solitons and form \( \tan \sum_{i=1}^{3} \left( \frac{\phi_i}{4} \right) \).

This is given by the formula above with \( u_{12} = u_{13} = u_{23} = 1 \) and represents the profile of three noninteracting solitons. The three-soliton solution is now obtained by «switching on» the interactions between pairs of solitons by introducing the invariant multipliers \( u_{ij}^4 < 1 \) into each product \( \tan \left( \frac{\phi_i}{4} \right) \tan \left( \frac{\phi_j}{4} \right) \).

The construction given by Bowtell and Stuart can clearly be extended to \( N \) solitons for general \( N \). Although these authors pointed this out, they did not provide a proof that the resulting formula was a \( N \)-soliton solution. Since it is clearly a much simpler representation than the Hirota formula, we feel that it is worthwhile to present a direct proof of its validity and to extend it to the general case where the multisoliton contains solitons, antisolitons and breathers in any combination.

In an earlier paper (2), we presented a preliminary version of the proof of the conjecture for the case of multisolitons which are a combination of solitons and antisolitons only. In this paper we present a more comprehensive treatment by extending the representation to include breathers. Furthermore, we also remedy certain deficiencies, both in form and content, which occurred in the

earlier paper. Since the formula is constructed by taking the tangent of a sum of elementary components (to be defined later for the case of breathers), we refer to it as a nonlinear superposition formula. Our proof of this nonlinear superposition formula is based on the inverse-scattering result.

Although the interpretation given by the authors of (4) describes the multisoliton solution as a set of pairwise interactions, it still has one major drawback when compared with the corresponding particle problem. This is that the elementary components in a multisoliton solution can only be identified in the noninteracting limits \( t \to \pm \infty \). For all finite times their identities are lost in the nonlinearity of the multisoliton formula. In order to identify these components throughout the interaction process, it is necessary to have the solution in a form which is manifestly separable into \( N \) component functions at all finite times and not just asymptotically. Now one way of achieving this objective is to attempt to pull back the linear superposition form from \( |t| = \infty \) to finite \( t \). We have done this via the nonlinear superposition formula and the resulting linear superposition is in terms of basis functions which are the eigenvalues of a symmetric matrix constructed once again from the elementary solutions of the SGE. The two-component case can be solved explicitly and it turns out that the superposition is in terms of accelerating or oscillating kinks which asymptotically reduce to the usual constant speed kinks. Numerical solutions for more than two components (4) indicate that this is also the case in general. Hence this basis gives a faithful representation of the multisoliton as a set of interacting «solitons» for any finite time as well as asymptotically. Furthermore, the existence of this superposition also reinforces the idea that solitons behave as classical particles. Finally, we wish to emphasize that the linear superposition we have obtained is different from the well-known one in which a multisoliton solution is expanded in terms of the eigenfunctions of the corresponding inverse-scattering problem.

The plan of the rest of the paper is as follows. In sect. 2 we introduce the elementary solutions of the SGE which we use in our construction. In sect. 3 we obtain both the nonlinear and the linear superposition formulae for a general multisoliton and in sect. 4 we discuss some of the properties of the basis functions of the linear superposition formula. Our concluding remarks are contained in sect. 5.

2. – General formalism.

We consider the SGE

\[
\ddot{\varphi}_{xx} - \dot{\varphi}_x = \sin \varphi,
\]

where \( \phi \) is usually a real-valued map from \( \mathbb{R}^2 \) to \( \mathbb{R} \). The real irreducible elements of the soliton sector of (1) are the solitary waves and breather solutions, and we use the following representations for these components.

The solitary waves in their rest frame are given by

\[
\phi = 4 \tan^{-1}(\varepsilon \exp[x]),
\]

where \( \varepsilon = +1 \) corresponds to the soliton with range \((0, 2\pi)\) and \( \varepsilon = -1 \) to the antisoliton with range \((-2\pi, 0)\).

The breather is a periodic wave with a solitary-wave envelope and the property that both the wave and its envelope travel at the same speed. As a consequence it has a rest frame in which it is a standing wave and in this frame it is given by the expression

\[
\phi = 4 \tan^{-1}[(\sigma/\omega) \text{sech} xx \sin \omega t],
\]

where \( \sigma = (1 - \omega^2)^{1/2} \) and \((2\pi/\omega) \in (2\pi, \infty)\) is the period of oscillation of the standing wave.

Remark. In (4) we represented the solitary-wave solutions by the formula \( \phi = 4 \tan^{-1}(\exp[xx]) \), which puts both the soliton and the antisoliton in the range \((0,2\pi)\). Our reason for going over to the expression given in (2), which puts the soliton and antisoliton in different ranges, is that it has the following advantages over the conventional form. Firstly, it leads to a uniform treatment of the coupling coefficients in both the nonlinear and the linear superpositions and, secondly, it leads to real functions in the linear superposition formula.

The rest frame solutions (2) and (3) can be transformed to a general frame by making the usual Poincaré transformation

\[
x \rightarrow \gamma(x - ut) + \alpha, \quad t \rightarrow \gamma(t - ux) - u\alpha,
\]

where \( \gamma = (1 - u^2)^{-1/2} \) and \(|u| < 1\). In this frame these solutions move with speed \( u \) and have a phase \( \alpha \).

It is clear from (3) that the breather has no real separable asymptotic from in the limits \( t \to \pm \infty \). However, unlike the solitary wave, it is not considered to be a structureless excitation of the sine-Gordon field. The usual interpretation of the breather is that it is a composite solution which is a bound state of a soliton and an antisoliton. The only way in which we can see this structure explicitly is to introduce complex components, i.e. to construct the breather from complex elementary solutions. This is done as follows:

Using the construction given in (4), we note that the soliton-antisoliton
scattering state can be written in the form

\[
\operatorname{tg}(\phi/4) = \frac{\operatorname{tg}(\phi_1/4) + \operatorname{tg}(\phi_2/4)}{1 - u_{12}^2 \operatorname{tg}(\phi_1/4) \operatorname{tg}(\phi_2/4)},
\]

where \(\operatorname{tg}(\phi_1/4) = \exp[\gamma_1(x - u_1 t) + \alpha_1]\) is a soliton moving with speed \(u_1\) and phase \(\alpha_1\), \(\operatorname{tg}(\phi_2/4) = -\exp[\gamma_2(x - u_2 t) + \alpha_2]\) is an antisoliton moving with speed \(u_2\) and phase \(\alpha_2\), and \(|u_{12}|\) is their common speed in the centre of velocity frame.

We now consider the breather moving with speed \(u\) and rewrite it like the scattering state (4). It is possible to do this and the specific form we use is

\[
\operatorname{tg}(\phi/4) = \frac{i \exp[Z] - i \exp[\overline{Z}]}{1 + (\omega/\sigma)^2 \exp[Z + \overline{Z}]},
\]

where \(Z = \Gamma(x - Ut) + \alpha, \ U = (\sigma u + i \omega)/(\sigma + i \omega u), \ \Gamma = \gamma(\sigma + i \omega u)\) and \(\alpha \in \mathbb{C}\). It is easy to verify that \(\Gamma^2(1 - U^2) = 1\) and hence that

\[
\phi = 4 \operatorname{tg}^{-1}(i \exp[Z])
\]

and its conjugate \(\overline{\phi}\) are solutions of (1). Furthermore, we note that, when \(u = \omega = \alpha = 0\), then \(\exp[Z]\) reduces to \(\exp[x]\) and \(-\exp[Z]\) to \(-\exp[x]\). We therefore define \(\phi\) to be a complex soliton and, consequently, \(\overline{\phi}\) is a complex antisoliton. This now allows us to interpret the breather as a nonlinear superposition of the complex elementary solutions \(\phi\) and \(\overline{\phi}\).

To summarize, in this section we have introduced four elementary solutions of the SGE. These are the real soliton and antisoliton defined by eq. (2) and their complex counterparts defined by (6). These elementary solutions form the basis for our construction of multisolitons in the next section.

3. – Construction of the multisolitons.

We begin by defining notation and organizing the input data for our construction. For the real argument we let

\[
X = \gamma(x - ut) + \alpha,
\]

where \(u \in (-1, 1), \ \gamma = (1 - u^2)^{-1/2}\) and \(\alpha \in \mathbb{R}\). The complex argument \(Z\) was defined earlier (under eq. (5)) in terms of the complex parameters \(\Gamma\) and \(U\). However, if we let \(T = \gamma(t - u\omega)\) and use the definition of \(X\), then it is easy to see that

\[
Z = \sigma X - i \omega T + \alpha, \quad \alpha \in \mathbb{C},
\]
and so $Z$ is expressed entirely in terms of real parameters except for the phase $\alpha$. This is the form of $Z$ we use in our constructions.

The set of real soliton functions $s_k : \mathbb{R}^2 \to (0, 2\pi)$ is given by

$$S = \{ s_k = 4 \operatorname{tg}^{-1}(\exp[X_k]); u_j \neq u_k \}.$$

This is a set of two-parameter functions where the parameters are $u_k$ and $\alpha_k$. Given any $s_k \in S$, we define the functions

$$v_k = e_k s_k = 4 \operatorname{tg}^{-1}(e_k \exp[X_k]),$$

where $e_k = \pm 1$.

Thus, $v_k$ is a soliton if $e_k = +1$ and an antisoliton if $e_k = -1$. We now let $k = 1, \ldots, N - 2m$ to obtain a multiplet containing $(N - 2m)$ solitons and antisolitons moving with distinct velocities and denote this by the ordered set $(v_1, \ldots, v_{N - 2m})$.

The set of complex soliton functions $w_k : \mathbb{R}^2 \to \mathbb{C}$ is given by

$$S_c = \{ w_k = 4 \operatorname{tg}^{-1}(i \exp[Z_k]), U_k \notin R, U_j \neq U_k \}.$$

Note $U_j \neq U_k$ allows $u_j = u_k$ if $w_j = w_k$ and $w_j = w_k$ if $u_j = u_k$. We now choose a multiplet of complex solitons and antisolitons. Since our concern is only with real solutions, we require that these multiplets contain only conjugate pairs of the complex solutions. Thus, we define the ordered set $(w_k, \overline{w}_k)_{k=1}^{m} = (w_1, \overline{w}_1, \ldots, w_m, \overline{w}_m)$, which is a multiplet containing $m$ conjugate pairs of elementary complex solutions.

Now let $\zeta_{2k-1} = w_k, \zeta_{2k} = \overline{w}_k, 1 \leq k \leq m$, and $\zeta_k = v_k, 2m + 1 \leq k \leq N$. Then $(\zeta_k)_{k=1}^{N} = (\zeta_1, \ldots, \zeta_N)$ is a general multiplet of $N$ elementary components containing $(N - 2m)$ real solitons and antisolitons and $m$ conjugate pairs of complex solitons and antisolitons.

Note: going over to the $\zeta$'s merely involves a relabelling of the $v$'s and is done in order to eliminate confusion in the parameter labels.

The construction of the multisoliton from the elementary solutions uses Lorentz invariant scalars which act as coupling constants between pairs of components. From (6) we know that for real components they are given by $u_{rs}^2 \in (0, 1)$, where

$$u_{rs} = \frac{\gamma_r \gamma_s (u_r - u_s)}{1 + \gamma_r \gamma_s (1 - u_r u_s)}.$$

The extension to include complex components is that, if the $r$-th component is complex, then $u_r, \gamma_r$ in (9) are replaced by $U_r, \Gamma_r$. These couplings are complex except when $r, s$ represent the $k$-th conjugate pair $(w_k, \overline{w}_k)$ in which case $u_{rs} = -(\omega_k / \omega_k)^2 \in (-\infty, 0)$. 


This completes the organization of the input data. We state now and prove the main theorems.

3.1. **Nonlinear superposition formula.** – We begin with the construction of the nonlinear superposition formula.

**Theorem 1.** Let \((\phi_1, \ldots, \phi_N)\) be a general multiplet as defined above and let 
\(\sigma_N = \sum_{k=1}^{N} \phi_k\) be the linear superposition of the \(N\) elementary components of the multiplet. In the expansion of \(\tan(\sigma_N/4)\) in terms of the components \(\tan(\phi_k/4)\), \(1 \leq k \leq N\), make the transformation \(\tan(\phi_j/4)\tan(\phi_k/4) = u_{jk} \tan(\phi_k/4)\) for every pair \(1 \leq j < k \leq N\), where \(u_{jk}\) is specified in (9), and call the transformed function \(\tan(\varphi_N/4)\). Then \(\varphi_N\) is a real multisoliton of the SGE containing \((N - 2m)\) solitons and antisolitons and \(m\) breathers. □

Before proving the theorem, we illustrate the construction defined in the theorem by the following three-component example:

**Example.** (Soliton breather) We input the multiplet \((\phi_1, \phi_2, \phi_3)\) where \(\phi_1\) and \(\phi_2\) are a conjugate pair of complex elementary solutions and \(\phi_3\) is real. Then, using the notation \(e_k = \tan(\phi_k/4)\), we have \(\sigma_3 = \phi_1 + \phi_2 + \phi_3\) and

\[
\begin{align*}
e_1 &= \tan(\phi_1/4) = i \exp[Z_1], \\
e_2 &= \tan(\phi_2/4) = -i \exp[\overline{Z}_1], \\
e_3 &= \tan(\phi_3/4) = \exp[X_3].
\end{align*}
\]

The coupling coefficients are \(u_{12} = i\omega/\sigma, u_{13} = u_{23} = \delta\), where

\[
\delta = \frac{I_1 \gamma_3 (U_1 - u_3)}{1 + I_1 \gamma_3 (1 - U_1 u_3)}.
\]

Implementing the construction defined in the theorem gives

\[
\tan(\varphi_3/4) = \frac{e_1 + e_2 + e_3 - |\delta|^4 (\omega^2/\sigma^2) e_1 e_2 e_3}{1 - (-\omega^2/\sigma^2) e_1 e_2 - \delta^2 e_1 e_3 - \delta^2 e_2 e_3}.
\]

Since \(e_3 = \overline{e}_1\), this expression is clearly real and, writing \(\varphi = r \exp[i\theta]\), reduces to the form

\[
\tan(\varphi_3/4) = \frac{2 \exp[\omega X_1] \sin \omega T_1 + \exp[X_3] + (r^4 \omega^2/\sigma^2) \exp[2\omega X_1 + X_3]}{1 + (\omega/\sigma)^2 \exp[2\omega X_1] - 2r^2 \exp[\omega X_1 + X_3] \sin(\omega T_1 - 2\theta)}.
\]
where the phase of $Z_1$ has been absorbed into $X_1$ and $T_1$. Now in the rest frames $(\xi_1, \eta_1)$ of $X_1$ and $(\xi_3, \eta_3)$ of $X_3$ it is easy to show that, as $\eta_1$ and $\eta_3 \to \pm \infty$, then, modulo phases,

$$
\Psi_3 \to 4 \mathrm{tg}^{-1} [\mathrm{sech} \, \xi_1 \sin \omega \eta_1],
$$

$$
\Psi_3 \to 4 \mathrm{tg}^{-1} [\exp [\xi_3]],
$$

respectively. Comparing with eqs. (2) and (3) we see that $\Psi_3$ is a multisoliton containing one soliton and one breather.

**Proof of the theorem.** The formula for a real multisoliton solution obtained by using the inverse-scattering method (7) is as follows:

\begin{equation}
\Psi_N = -4 \mathrm{tg}^{-1} \Delta_N,
\end{equation}

where

$$
\Delta_N = \frac{\text{Im} \, \det (I - iM)}{\text{Re} \, \det (I - iM)}
$$

and $M$ is the $N \times N$ matrix whose elements are defined in terms of parameters $a_k$ by

\begin{equation}
m_{jk} = \frac{2 \delta_k \alpha_k}{a_j + a_k} \exp \left[ \frac{1}{2} (a_j + a_k) \xi + (\gamma / a_k) + \alpha_k \right]
\end{equation}

with $\xi = \frac{1}{2} (x - t)$, $\eta = \frac{1}{2} (x + t)$. The parameters can be either real or complex, but if complex they must occur in conjugate pairs. The factor $\delta_k$ is just a phase which is included outside the exponential in order to deal with our representation of the elementary solutions. The input $(\varphi_\mu)_{\nu=1}^N$ is defined in terms of the $N$ real parameters $(u_k, \omega_k)_{k=1}^m$ and $(u_k)_{k=2m+1}^N$ and these are related to the inverse-scattering parameters $a_\nu$ as follows:

$$
a_{2k-1} = \gamma_k (1 + u_k)(\sigma_k + i\omega_k) = a_{2k}, \quad 1 \leq k \leq m,
$$

$$
a_k = \gamma_k (1 + u_k), \quad 2m + 1 \leq k \leq N.
$$

The values of $\delta$ are given by $\delta_{2k-1} = i$, $\delta_{2k} = -i$, $1 \leq k \leq m$ and $\delta_k = \varepsilon_k$, $2m + 1 \leq k \leq N$.

By expanding the determinant in $\Delta_N$, the inverse-scattering formula (10a) can

be put in the form

$$
\sum_{r=0}^{n_1} (-1)^r \mu_{2r+1} \over 1 + \sum_{r=1}^{n_2} (-1)^r \mu_{2r}.
$$

where $\mu_r$ is the sum of the principal minors of order $r$ of $\det M$ and $n_1 = [(1/2)(N-1)]$, $n_2 = [(1/2)N]$ with $[.]$ denoting integer part of. It is clear that the principal minors of $M$ have the same form as the determinant of $M$ and hence an explicit formula for $\det M$ will also determine the $\mu_r$. Now $M$ factorizes as follows:

$$
M = B(\xi) A b(\xi) C(\eta),
$$

where $A$ is the $N \times N$ matrix with elements $a_{jk} = 2a_k/(a_j + a_k)$ and $B$, $C$ are diagonal matrices with elements $b_{kk} = \exp [(1/2) a_k \xi]$, $c_{kk} = \delta_k \exp [\eta_k/a_k + \alpha_k]$, respectively. Thus

$$
\det M = \det A \prod_{k=1}^N \delta_k a_k \exp \left[ a_k \xi + \eta/a_k + \alpha_k \right] = \det A \prod_{k=1}^N a_k \tan \left( \phi_k/4 \right),
$$

since $a_k \xi + \eta/a_k + \alpha_k$ is equal to $X_k$ or $Z_k$ in laboratory coordinates.

Turning now to $\det A$ we write

$$
\det A \prod_{k=1}^N a_k = \det H_N,
$$

where $H_N$ is the symmetric $N \times N$ matrix with elements $h_{kj} = 2(a_ja_k)^{1/2}/(a_j + a_k)$. Straightforward elementary row operations and the extraction of row and column factors leads to the following iterative function for $H_N$:

$$
\det H_N = \det H_{N-1} \prod_{k=1}^{N-1} \left( \frac{a_N - a_k}{a_N + a_k} \right)^2.
$$

Noting that $((a_j - a_k)/(a_j + a_k))^2 = u_{jk}^2$, then successive applications of this result give

$$
\det H_N = \prod_{j=1}^{N-1} \prod_{k=j+1}^N u_{jk}^2.
$$

Hence

$$
\det M = \prod_{j=1}^{N-1} \prod_{k=j+1}^N u_{jk}^2 \prod_{n=1}^N \tan \left( \phi_n/4 \right).
$$
We can now form the sums of the principal minors of $M$ as follows:

\[ \mu_1 = \sum_{n=1}^{N} \tan \left( \frac{\phi_n}{4} \right), \quad \mu_r = \sum_{n=1}^{r-1} \prod_{j=k+1}^{r} u_{jk} \prod_{n=1}^{r} \tan \left( \frac{\phi_n}{4} \right) \quad (r > 1), \]

where the expression behind the summation sign in $\mu_r$ is the first principal minor and all other principal minors are obtained by taking all combinations of $r$ integers from $1, \ldots, N$ in place of $1, \ldots, r$. Clearly, by using (12) for the $\mu_r$ in (11) with all multipliers $u_{jk} = 1$, the expression (11) for the multisoliton $\Psi_N$ reduces to

\[ \tan \left[ \sum_{k=1}^{N} \left( \frac{\phi_k}{4} \right) \right]. \]

Hence, with the multipliers included, we have the form stated in the theorem.

This completes the presentation of the nonlinear superposition formula for sine-Gordon multisolitons.

3'2. Linear superposition formula. – We now turn to the representation of a multisoliton as a linear superposition.

**Theorem 2.** Let $(\phi_1, \ldots, \phi_N)$ be the multiplet of theorem 1 and let $e_k = \tan \left( \frac{\phi_k}{4} \right)$. Consider the polynomial

\[ P(f) = \sum_{k=0}^{N} (-1)^r A_r f^{N-r}, \]

where $A_0 = 1$ and the other coefficients $A_r$ are constructed from the elements of the multiplet such that $A_r = \mu_r$, where the $\mu_r$ have already been defined in eq. (12). Then the multisoliton $\Psi_N$ of theorem 1 can be represented as follows:

\[ \Psi_N = \sum_{k=1}^{N} g_k(x, t), \]

where $g_k(x, t) = 4 \tan^{-1} [f_k(x, t)]$ and the $f_k$ are the roots of $P(f)$. \( \Box \)

**Proof.** Comparing the expansion $\tan \left( \sum_{k=1}^{N} \tan^{-1} f_k \right)$ with the expression for $\Psi_N$ given in theorem 1, it follows from eq. (11) that they will be the same if the $f_k$ satisfy the identities

\[ \sum_{k=1}^{N} f_k = \mu_1, \quad \sum_{i=1}^{r} f_i = \mu_r \quad (r > 1), \]

where the summation is over all combinations of $r$ integers from $1, \ldots, N$ in place.
of 1, ... , r. Using (12) for the $\mu_r$, it follows that the $f_k$ are the roots of the polynomial $P(f)$. □

To illustrate the theorem we present the polynomial for the three-component example considered after theorem 1, i.e. a soliton breather.

**Example.** Using the notation of the previous example for the explicit forms of the quantities involved we have

$$A_1 = 2 \exp[zX_1] \sin(oT_1) + \exp[X_3],$$
$$A_2 = -\left(\frac{o}{2}\right)^2 \exp[2zX_1] + 2r^2 \exp[zX_1 + X_3] \sin(oT_1 - 2\theta),$$
$$A_3 = -\left(\frac{r^4}{4}\right)^2 \exp[2zX_1 + X_3],$$

and hence the $f_k$, $k = 1, 2, 3$, are the roots of the cubic

$$f^3 - A_1 f^2 + A_2 f - A_3 = 0.$$

We look now at the asymptotic properties of the functions $g_k(x, t)$. Since $\mathcal{V}_N$ is a multisoliton, it separates asymptotically into its irreducible elements, i.e. solitons, antisolitons and breathers. The following corollary shows that these irreducible elements are also the asymptotic forms of the functions $g_k(x, t)$ when considered individually or in pairs.

**Corollary 2.1.** Let $\varphi$ be a soliton or antisoliton and $\varphi_B$ a breather of the multisoliton $\mathcal{V}_N$. Then

i) there exists a $g$ whose asymptotic form is $\varphi$,

ii) there exists a pair of $g$’s whose asymptotic form is $\varphi_B$. □

**Remark.** This does not imply that each $g$ is asymptotic to the same $\varphi$ or $\varphi_B$ and at $t = \infty$. For example, a particular $g$ function may be a soliton at $t = -\infty$ and part of a breather at $t = \infty$. (Cf. the two-soliton example in sect. 4.)

**Proof.** Relabel the multiplet $(\varphi_m)_{k=1}^N$ so that the elements are ordered in terms of increasing velocity, i.e. $u_k < u_{k+1}$. Choose an element $\varphi_n$ and rewrite the polynomial (13) in the rest frame $(\xi, \eta)$ of $\varphi_n$ using the Lorentz velocity transformation: $u_k \rightarrow u_{kn} = (u_k - u_n)/(1 - u_k u_n), \eta_k \rightarrow I_{kn} = (1 - u_{kn}^2)^{-1/2}$. Note that, if $\varphi_n$ is complex, then it has the same rest frame as its conjugate, in which case we are considering the pair of elements $(\varphi_n, \varphi_{n+1} = \bar{\varphi}_n)$.

In this frame $(\xi, \eta)$ the exponentials $e_k$ in the coefficients $A_k$ of the polynomial have the following behaviour as $\eta \rightarrow \pm \infty$: $e_k \rightarrow 0$ if $k > n$ and $e_k \rightarrow \pm \infty$ if $k < n$. 
Dividing the polynomial by \( p = \prod_{k=1}^{n-1} e_k \) and putting \( B_r = A_r/p \) gives the expression

\[
(15) \quad \sum_{r=0}^{N} (-1)^r B_r f^{N-r}.
\]

Letting \( \eta \to \infty \) and assuming that at least one of the roots of the polynomial is finite in this limit gives two asymptotic forms:

i) If \( \varphi_n \) is a real component, then polynomial (15) reduces to

\[
f^{N-n} (B_{n-1}^* f - B_n^*),
\]

where \( B_r^* = \lim_{\omega \to \infty} B_r \). Hence there is a unique finite root \( E_n = e_n \prod_{j=1}^{n-1} u_j^* \) and, thus, there exists a \( g \) whose asymptotic form is \( 4 \tan^{-1} E_n \), which is \( \varphi_n \) modulo phases.

ii) If \( \varphi_n \) is complex, then the asymptotic form of polynomial (15) is

\[
f^{N-n-1} (B_{n-1}^* f^2 - B_n^* f + B_{n+1}^*)
\]

and its nonzero finite roots are those of the quadratic

\[
f^2 - (E_n + E_n^*) - (\omega_n/\varphi_n)^2 E_n E_n^* = 0.
\]

Clearly these roots are real and of opposite sign and define the asymptotic forms of two \( g \) functions. Adding these functions gives

\[
4 \tan^{-1} \left[ \frac{E_n + E_n^*}{1 - (\omega_n/\varphi_n)^2 E_n E_n^*} \right]
\]

and comparing this expression with eq. (5) we see that this is \( \varphi_B \) modulo phases.

The case \( \eta \to -\infty \) can be dealt with in a similar manner. This completes the proof of the corollary. \( \square \)

4. – Properties of the functions in the linear formula.

The functions of the linear superposition formula provide us with a basis for expansions in the soliton sector of the SGE. The computation of this basis is well defined and the functions themselves have the right asymptotic behaviour. However, questions related to uniqueness and universality, with a view to more general expansions, require a much more detailed exploration of the properties of this basis and in this section we present some results in this direction.
Firstly, we note that the coefficients in the polynomial (13) are always real and that the roots, i.e., our basis functions, are analytic functions of their arguments. Secondly, the polynomial is the characteristic polynomial of a well-defined matrix as the following lemma shows:

**Lemma 1.** The functions $f_k(x, t)$ are the eigenvalues of the symmetric $N \times N$ matrix $Q$ whose elements are defined as follows:

$$q_{jk} = [\delta_j \delta_k (1 - u_{jk}^2)]^{1/2} \exp \left[ \frac{1}{2} (Y_j + Y_k) \right] \quad (j \neq k),$$

$$q_{kk} = \delta_k \exp \{ Y_k \},$$

where $\delta_{2k-1} = i = \overline{\delta_{2k}}, \quad Y_{2k-1} = Z_k = \overline{Y}_{2k}$ for $1 \leq k \leq m$, and $\delta_k = \varepsilon_k, \ Y_k = X_k$ when $2m + 1 \leq k \leq N$. □

**Proof.** Expressing the elements of $Q$ in terms of the $a_k$ and $e_k$ we have that $q_{jk} = 2(a_k a^*_k e_j e^*_k)^{1/2}(e_j + a_k)$. Thus $Q$ can be expressed as the matrix product $Q = \text{diag} (\sqrt{e_1}, ..., \sqrt{e_N}) H_N \text{diag} (\sqrt{e_1}, ..., \sqrt{e_N})$, where $H_N$ is the matrix defined in the proof of theorem 1. Hence

$$\det Q = \prod_{k=1}^{N} e_k \prod_{j=1}^{N-1} \prod_{k=j+1}^{N} u_{jk}^2.$$

The symmetry of $Q$ implies that the principal minors of $Q$ have the same form as $\det Q$ and hence it is easy to see, from eq. (13) defining $A_r$ above, that $A_r$ is the sum of the principal minors of order $r$ of $Q$. Thus the polynomial in $f$ is the characteristic polynomial of $Q$, i.e., the $f_k(x, t)$ are the eigenvalues of $Q$. □

When the multisoliton contains only solitons or only antisolitons, then the $f_k(x, t)$ are described by the following theorem:

**Theorem 3.** If the multisoliton contains only solitons or only antisolitons, then the $f_k(x, t)$ are real functions which are positive in the case of solitons and negative for antisolitons. □

**Proof.** In each of these cases $\delta_j \delta_k = e_j e_k = +1$ and $u_{jk}^2 \in (0, 1)$ for all $j, k$. Thus, $Q$ is a real symmetric matrix and consequently all its eigenvalues are real. Applying Descartes’ rule of signs to the polynomial (13) satisfied by the $f_k$, $1 \leq k \leq N$, shows that these functions are all positive when $e_k = +1$ for all $k$ and all negative when $e_k = -1$ for all $k$. □

When the multisoliton contains both solitons and antisolitons (either real or complex), then the matrix $Q$ is no longer real and neither is it Hermitian. Thus
we cannot immediately deduce that its eigenvalues are real. However, let us consider the following examples.

**Example (Soliton-antisoliton).** In this case \( \varepsilon_1 = - \varepsilon_2 = 1 \). Using \( \alpha = (1 - u_{12}^2)^{1/2} \), then

\[
Q = \begin{bmatrix}
\exp[X_1] & i\alpha \exp\left[ \frac{1}{2} (X_1 + X_2) \right] \\
i\alpha \exp\left[ \frac{1}{2} (X_1 + X_2) \right] & - \exp[X_2]
\end{bmatrix},
\]

so that the eigenvalues of \( Q \) are given by the roots of the quadratic equation

\[
f^2 - (e^{X_1} - e^{X_2})f - u_{12}^2 e^{X_1 + X_2} = 0.
\]

These roots real and of opposite sign.

Note: The representation for antisolitons used in (\(^8\)) (see remark after eq. (3)) leads to the quadratic

\[
f^2 - (e^{X_1} + e^{-X_2})f + (1/u_{12})^2 e^{X_1 - X_2} = 0.
\]

This has complex conjugate roots and explains the reasons for changing our representation to that given in eq. (2).

**Example (Breather).** Using the notation introduced earlier, the polynomial in this case is the quadratic equation

\[
f^2 - (2 \exp[\sigma X] \sin \omega T)f - (\omega/\sigma)^2 \exp[2\sigma X] = 0,
\]

which has two real roots with opposite signs.

The above examples for \( N = 2 \), together with numerical evidence (\(^9\)) for the cases of \( N = 3, 4 \) and \( 5 \), lead us to make the following conjecture on the nature of the eigenvalues for a general multisoliton:

**Conjecture.** Let the multisoliton contain \( n_1 \) solitons, \( n_2 \) antisolitons and \( n_3 \) breathers, i.e. \( n_1 + n_2 + 2n_3 = N \). Then the corresponding matrix has \( n_1 + n_3 \) real positive eigenvalues and \( n_2 + n_3 \) real negative eigenvalues. \( \Box \)

We conclude this section on the properties of the \( f \) functions by looking at their structural forms in the case \( N = 2 \) which is explicitly solvable. These forms are most transparent in the centre-of-velocity frame for the scattering states and in the rest frame of the bound state, i.e. the breather. In these special frames it
turns out that the \( g \) functions are accelerating kinks for the scattering states and oscillating kinks for the breather. The details are as follows:

The general two-component multisoliton is given by

\[
 \Psi_2 = 4 \tan^{-1} f_1 + 4 \tan^{-1} f_2,
\]

where \( f_1 = \exp \left[ \alpha (x + \beta(t)) \right] \), \( f_2 = \epsilon \exp \left[ \alpha (x - \beta(t)) \right] \) and the specific cases are

i) soliton-soliton: \( \epsilon = +1 \), \( \alpha = y \) and

\[
 \beta(t) = \left( \frac{1}{\gamma} \right) \ln \left\{ \cosh \gamma u t + (\cosh^2 \gamma u t - u^2)^{1/2} / u \right\},
\]

ii) soliton-antisoliton: \( \epsilon = -1 \), \( \alpha = y \) and

\[
 \beta(t) = \left( \frac{1}{\gamma} \right) \ln \left\{ \sinh \gamma u t + (\sinh^2 \gamma u t + u^2)^{1/2} / u \right\},
\]

iii) breather: \( \epsilon = -1 \), \( \alpha = \sigma \) and

\[
 \beta(t) = \left( \frac{1}{\sigma} \right) \ln \left\{ \sin \omega t + (\sin^2 \omega t + (\omega / \sigma)^2)^{1/2} / (\sigma / \omega) \right\}.
\]

The expressions in a general frame are obtained from the above by Lorentz transformation. However, it is clear that in this case we lose the rigid motion of the kinks.

5. – Concluding remarks.

In this paper we have constructed the general real multisoliton of the SGE as a nonlinear superposition of its elementary real and complex solutions. By a general multisoliton we mean one containing any combination of solitons and breathers and, in this instance, the inclusion of breathers necessitates the introduction of complex elementary solutions. Among the various methods of constructing a multisoliton\(^{(1)}\) we feel that the method presented here is the simplest. Furthermore, it also leads to a natural interpretation of the multisoliton as a many-body system in which the individual components interact in pairs, thus emphasizing the particle-like behaviour of solitons\(^{(1)}\). The nonlinear superposition construction leads in a straightforward manner to a linear superposition formula in which the basis functions are the eigenvalues of a symmetric matrix constructed once again from the elementary solutions of the SGE. In the case \( N = 2 \), we showed explicitly that these basis functions are

accelerating kinks for the scattering states and oscillating kinks for the breather. Unfortunately, we have not been able to extend this analysis to the cases \( N > 2 \), but numerical computations (1) of the eigenvalues indicate that the same behaviour persists, \( i.e. \) that the basis functions are either accelerating or oscillating kinks. These observations reinforce our previous suggestions that the identification of individual solitons in the multisoliton at any finite time, as well as asymptotically, can only be made in a consistent manner in terms of these accelerating and oscillating kinks. We are currently trying to establish that the basis functions have the structural form

\[
4 \tan^{-1} \{ \exp [\alpha(x, t)(x - \beta(x, t))] \}
\]

and to determine the behaviour of the functions \( \alpha(x, t) \) and \( \beta(x, t) \). Finally, we point out that our constructions are realizations of a map from the solutions of the linearized SGE, \( i.e. \)

\[
\phi_{xx} - \phi_{tt} = \phi,
\]

to the SGE itself.

\[\text{RIASSUNTO (*)}\]

Si presentano formule di sovrapposizione non lineari e lineari per le soluzioni del multisolitone dell'equazione di sine-Gordon. Queste formule sono composte per il caso generale, cioè per multisolitoni che contengono qualsiasi combinazione di solitoni, antisolitoni e «breathers». La sovrapposizione non lineare si costruisce da questi componenti, mentre la sovrapposizione lineare è in termini di kink di accelerazione e oscillazione. Si afferma che questi ultimi forniscono un'interpretazione più coerente di una soluzione del multisolitone come un'interazione tra i suoi componenti asintotici.

(*) Traduzione a cura della Redazione.

\[\text{Формулы суперпозиции для многосолитонных решений уравнения син-Гордона.}\]

\[\text{Резюме (*) - Мы предлагаем формулу нелинейной и линейной суперпозиции для многосолитонных решений уравнения син-Гордона. Эти формулы конструируются в общем случае, \( t.e. \) для многосолитонных решений, которые содержат любую комбинацию солитонов, антисолитонов и брежеров. Из этих компонент конструируется нелинейная суперпозиция, тогда как линейная суперпозиция конструируется в терминах ускоренных и осциллирующих частиподобных решений. Мы показываем, что предложенная линейная суперпозиция обеспечивает более последовательную интерпретацию многосолитонного решения, как взаимодействие между его асимптотическими компонентами.}\]

(*) Переведено редакцией.