Superposition Formulae for Multisolitons.

II. – The Modified Korteweg-de Vries Equation (*).

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Summary. — Following our work on the sine-Gordon equation we present nonlinear and linear superposition formulae for multisoliton solutions of the modified Korteweg-de Vries equation.

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1. – Introduction.

In a previous paper (1), hereafter referred to as I, we presented simple constructions of the multisoliton solutions of the sine-Gordon equation (SGE), firstly, as nonlinear combinations of elementary solutions, e.g., solitons, and, secondly, as linear superpositions of accelerating solitary waves, e.g., kinks. In this paper we extend these results to the modified Korteweg-de Vries equation (MKdV). Since the proofs of the main theorems are more or less the same as those given in I, we confine ourselves to stating the theorems and pointing out essential modifications where necessary. One significant difference which we mention at the outset concerns the structure of the breather solutions of these equations. Whereas the breather solution of the SGE has a unique rest frame, i.e. a frame in which the solution is a standing wave, this is not the case for the breather solution of the MKdV (2).

(*) To speed up publication, the authors of this paper have agreed to not receive the proofs for correction.

2. – General formalism.

We consider the MKdV equation

\[ \partial_t \phi + 6 \partial_x \phi_x + \partial_{xxx} \phi = 0, \]

where \( \phi \) is usually a real-valued map from \( \mathbb{R}^2 \) to \( \mathbb{R} \). The solitary wave solutions are given by

\[ \phi = \varepsilon a \text{sech} X = 2(\partial/\partial x) \tanh^{-1}(\varepsilon \exp[X]), \]

where \( X = a(x - a^2 t) + \alpha, \alpha \in \mathbb{R}, a \in \mathbb{R} \) and the usual convention is to refer to the solution as a soliton if \( \varepsilon = +1 \) and an antisoliton if \( \varepsilon = -1 \).

The breather solution of (1) in a general frame is given by

\[ \phi = 2 \frac{\partial}{\partial x} \tanh^{-1} \left[ \frac{\lambda \sin[\mu x - \mu (3\lambda^2 - \mu^2) t + \alpha]}{\mu \cosh[\lambda x - \lambda (\lambda^2 - 3\mu^2) t + \beta]} \right], \]

where \( \lambda, \mu, \alpha, \beta \in \mathbb{R} \) with \( \lambda, \mu \neq 0 \). It is clearly irreducible over the reals and, following the arguments developed in I, we shall need complex solitons and antisolitons in order to construct the breather components of a multisoliton. To this end we note that

\[ \phi_c = 2(\partial/\partial x) \tanh^{-1}(i \exp[Z]), \]

where \( Z = a(x - a^2 t) + \alpha \) and \( a, \alpha \in \mathbb{C} \), is a complex solution of (1) with the property that \( \exp[Z] \) reduces to \( \exp[X] \) when \( a \) and \( \alpha \) are both real. We therefore define \( \phi_c \) to be the complex soliton and its conjugate \( \bar{\phi}_c \) to be the complex antisoliton.

We refer to the four solutions defined above, i.e. the real soliton and antisoliton given by (2) and their complex counterparts (4), as the elementary solutions of the MKdV. They form the components for our constructions of the multisolitons given in the next section.

3. – Construction of the multisolitons.

Our constructions follow very closely those given in I for sine-Gordon multisolitons and will be presented in the form of two theorems. We start by defining notation and organizing our data. The argument for both the real and complex elementary solutions is \( a(x - a^2 t) + \alpha \). We call this \( X \) if both \( a \) and \( \alpha \) are real and \( Z \) when \( a \) is complex.

The set of real soliton functions is given by

\[ S = \{ s_k = 2(\partial/\partial x) \tanh^{-1}(\exp[X]); a_j \neq a_k \}, \]
where the condition $a_j \neq a_k$ implies that the elements of the set have distinct speeds. Using the notation $\partial = \partial / \partial x$, let $\partial v_k = \varepsilon_k s_k$. Then

$$v_k = 2 \tan^{-1}(\varepsilon_k \exp [X_k])$$

corresponds to a soliton for $\varepsilon_k = +1$ and an antisoliton for $\varepsilon_k = -1$. Hence, we represent a multiplet of $(N - 2m)$ real solitons and antisolitons by the ordered set $(v_1, \ldots, v_{N-2m})$.

The set of complex solitons is given by

$$S_c = \{ c_k = 2 \tan^{-1}(i \exp [Z_k]); a_j \neq a_k, a_k \notin \mathcal{R} \}.$$

In a similar manner to the above we let $c_k = \bar{\omega}_k$ and denote

$$w_k = 2 \tan^{-1}(i \exp [Z_k]).$$

To construct real multisolitons we need these complex components to enter in conjugate pairs and, restricting ourselves to this case, define a multiplet of $m$ conjugate pairs by the ordered set $(w_1, \bar{w}_1, \ldots, w_m, \bar{w}_m)$, $a_k \notin \mathcal{R}$. The condition that $a_k$ is not purely imaginary is necessary for the construction of breathers.

The general multiplet of $N$ elementary solutions containing $(N - 2m)$ real solitons and antisolitons and $m$ conjugate pairs of complex solitons and antisolitons is now defined by $(\varphi_k)_{k=1}^N$ and $(\bar{\varphi}_k)_{k=1}^N$, where $\varphi_{2k-1} = w_k$, $\varphi_{2k} = \bar{w}_k$, for $1 \leq k \leq m$, and $\varphi_k = v_k$ when $2m + 1 \leq k \leq N$.

The constants which couple pairs of elementary solutions in the superposition formulae are $u_{jk}^2$, where

$$(5) \quad u_{jk} = (a_j - a_k)/(a_j + a_k).$$

If $a_k$ is complex, we write $a_k = \lambda_k + i \mu_k$ and note that in the special case when $a_j = \bar{a}_k$, i.e., $j, k$ represent a conjugate pair $(w_k, \bar{w}_k)$, then $u_{jk}^2 = (-\mu_k/\lambda_k)^2$.

This completes the notation and we now state the main theorems.

**Theorem 1** (Nonlinear superposition). Let $(\varphi_k)_{k=1}^N \equiv (\bar{\varphi}_k)_{k=1}^N$ be a general multiplet of elementary solutions and let $\varphi_N = \sum_{k=1}^N \varphi_k$. In the expansion of $\tan(\varphi_N/2)$ in terms of the components $\tan((\varphi_j/2)$, $1 \leq k \leq N$, make the transformation $\tan(\varphi_j/2) \tan(\varphi_k/2) \to u_{jk}^2 \tan(\varphi_j/2) \tan(\varphi_k/2)$ for every pair $1 \leq j < k \leq N$, where $u_{jk}$ is specified by (5), and call the transformed function $\tan(\Phi_N/2)$. Then $\Phi_N = \Re \Phi_N$ is a real multisoliton of the MKdV containing $(N - 2m)$ solitons and antisolitons and $m$ breathers. □

**Example** (Soliton breather). Consider the multiplet $(\Re \varphi_1, \Re \varphi_2, \Re \varphi_3)$ of elementary solutions where $\varphi_1$ and $\varphi_2$ are a conjugate pair and $\varphi_3$ is real. Then,
using the notation \( e_k = \tg (\psi_k / 2) \), we have \( \sigma_3 = \psi_1 + \psi_2 + \psi_3 \) and

\[
e_1 = \tg (\psi_1 / 2) = i \exp[Z_1],
\]

\[
e_2 = \tg (\psi_2 / 2) = i \exp[Z_1],
\]

\[
e_3 = \tg (\psi_3 / 2) = i \exp[X_3].
\]

The coupling coefficients are \( u_{12}^2 = (\mu / \lambda)^2 \), \( u_{13}^2 = u_{23}^2 = \delta^2 \), where

\[
\delta = \frac{a_3 - a_1}{a_3 + a_1} = \frac{a - (\lambda + i \mu)}{a + (\lambda + i \mu)}.
\]

Implementing the construction defined in the theorem gives

\[
tg(Y_3/2) = \frac{e_1 + e_2 + e_3 - |\delta|^4 (-\mu^2 / \lambda^2) e_1 e_2 e_3}{1 - (-\mu^2 / \lambda^2) e_1 e_2 - \delta^2 e_1 e_3 - \delta^2 e_2 e_3}.
\]

Since \( e_3 = e_1 \), this expression is clearly real. We now let \( Z_1 = p + iq \) where \( p = \lambda x - \lambda (\chi^2 - 3 \mu^2) t + \Re x \), \( q = \mu x - \mu (3\lambda^2 - \mu^2) t + \Im x \) and write \( \delta = r \exp [i\theta] \). Then

\[
tg(Y_3/2) = \frac{\exp[X_3] - 2 \exp[p] \sin q + (r^4 \mu^2 / \lambda^2) \exp[2p + X_3]}{1 + (\mu^2 / \lambda^2) \exp[2p] + 2r^2 \exp[p + X_3] \sin(q + 2\theta)}
\]

and the multisoliton is given by \( \Phi_3 = \Theta Y_3 \).

To see that this does contain one soliton and one breather, we look at the asymptotic forms. Assuming that \( \lambda^2 - 3 \mu^2 - \alpha^2 \neq 0 \), then we have the following:

i) In the rest frame of \( X_3 \), i.e. \( \xi = x - \alpha^2 t \), \( \eta = t \), we have, modulo phases,

\[
\lim_{|\xi| \to \infty} Y_3 = 2 \tg^{-1}(\exp[\alpha \xi]),
\]

which gives the soliton.

ii) Now let \( \xi = x - (\lambda^2 - 3 \mu^2) t \), \( \eta = t \). This is the rest frame of \( p(x, t) \) and, from eq. (3), we see that this is the frame in which the envelope of the breather is stationary. Then, making use of the fact that \( Y_3 \) is only defined up to a constant and working modulo phases, we have

\[
\lim_{|\xi| \to \infty} Y_3 = 2 \tg^{-1} \left[ \frac{\lambda \sin q(\xi, \eta)}{\mu \cosh \lambda \xi} \right],
\]

which, from eq. (3), is the breather.
Proof of the theorem. The inverse-scattering method leads to the following form for multisolitons:\(^{(2, 3)}\):

\[ \Phi_N = 2\Theta \tan^{-1} \Delta_N, \]

where \( \Delta_N = \text{Im} \det (I - iM) / \text{Re} \det (I - iM) \) and \( M \) is the \( N \times N \) matrix with elements

\[ m_{jk} = \frac{2a_k \delta_{jk}}{a_j + a_k} \exp \left[ \frac{1}{2} (a_j + a_k) x - a_j t + a_k t \right]. \]

Complex parameters \( a_k \) must occur in conjugate pairs and the values of \( \delta_k \) are defined as follows: \( \delta_{2m-1} = i = \delta_{2m} \), for \( 1 \leq k \leq m \), and \( \delta_k = \delta_k \) when \( 2m + 1 \leq k \leq N \).

The proof now consists of showing that \( \Delta_N \) can be transformed to the expression for \( \tan (\pi \nu / 2) \) stated in the theorem and the details of this procedure follow those presented in I. \[ \square \]

THEOREM 2 (Linear superposition). Let \( (\varphi_k)_{k=1}^{N} = (\Theta \varphi_k)_{k=1}^{N} \) be the multiplet of theorem 1 and let \( e_k = \tan (\varphi_k / 2) \). Consider the polynomial

\[ P(f) = \sum_{r=0}^{N} (-1)^r A_r f^{N-r}, \]

where \( A_0 = 1 \) and the other coefficients \( A_r \), are constructed from the elements of the multiplet as follows:

\[ A_1 = \sum_{k=1}^{N} e_k, \quad A_r = \sum_{k=1}^{r} \prod_{k=1}^{r} e_k \prod_{j=k+1}^{r} u_{jk}^2 \quad (2 \leq r \leq N), \]

and the summation is over all combinations of \( r \) integers from 1, \ldots, \( N \) in place of 1, \ldots, \( r \). Then the multisoliton \( \varphi_N \) of theorem 1 can be represented as follows:

\[ \varphi_N = \sum_{k=1}^{N} g_k(x, t), \]

where \( g_k = 2\Theta \tan^{-1} [f_k(x, t)] \) and the \( f_k \) are the roots of \( P(f) \). \[ \square \]

Proof. The proof given in I can be used to show that \( \varphi_N = \sum_{k=1}^{N} 2 \tan^{-1} f_k \) and hence the theorem follows. \[ \square \]

As in the case of the SGE, we also have the following corollary:

**Corollary 2.1.** The asymptotic forms for the $g$ functions are the components of the multisoliton $\Phi_N$, *i.e.* solitons, antisolitons or breathers. □

4. — Properties of the functions in the linear formula.

Since the polynomial for the MKdV has exactly the same form as that of the SGE, it follows that all the results and conjectures made for the SGE also apply to the MKdV. Thus, the linear superposition is in terms of real analytic functions. We note, however, the following differences in detail: 1) there is no centre-of-velocity frame so the $N = 2$ examples are constructed in general frame; ii) the breather has no rest frame; iii) the basis functions have the form

$$\alpha(x, t) \operatorname{sech} \{ \beta(x, t) \} ,$$

where $x = \partial \beta / \partial x$. Based on numerical evidence, we conjecture that the basis functions are accelerating profiles which for each fixed $t$ are functionally close to the co-moving constant speed soliton. In particular, for each $t$, we expect these accelerating profiles to have the qualitative features of the sech function, *i.e.* one maximum, no zeroes and asymptotically null.

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**RIASSUNTO (*)

In seguito al nostro lavoro sull'equazione di sine-Gordon si presentano formule di sovrapposizione nonlineari e lineari per le soluzioni del multisolitone dell'equazione modificata di Korteweg-de Vries

(*) Traduzione a cura della Redazione.

**Формулы суперпозиции для многосолитонных решений. - II. Модифицированное уравнение Кортевега-де Фриза.**

**Резюме (*). — Следуя нашей предыдущей работе для уравнения син-Гордона, мы предлагаем формулы нелинейной и линейной суперпозиции для многосолитонных решений модифицированного уравнения Кортевега-де Фриза.**

(*) Переведено редакцией.